Lubell mass and induced partially ordered sets

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Abstract

We prove that for every partially ordered set P, there exists $c(P)$ such that every family $\mathcal{F}$ of subsets of $[n]$ ordered by inclusion and which contains no induced copy of P satisfies $\sum_{F\in\mathcal{F}} 1/\binom{n}{|F|} \leq c(P)$. This confirms a conjecture of Lu and Milans [9].

1 Introduction

A *partially ordered set*, or *poset*, is a set P equipped with some partial order relation $\leq$. A typical example of a partially ordered set is the hypercube on n vertices, namely $\mathcal{P}[n]$, the set of subsets of $\{1, 2, \dots, n\}$, equipped with the order relation $\leq$ so that for any two $A, B \in \mathcal{P}[n]$, $A \leq B$ if and only if $A \subseteq B$. A *chain* in a poset P is a collection $x_1, x_2, \dots, x_k$ of elements of P such that $x_1 < x_2 < \dots < x_k$. The *height* of P is the maximal size of a chain in P.

In this paper we consider a Turán-type question for posets: given a fixed poset P, what is the maximal size of a subset $\mathcal{F}$ of the hypercube which does not contain P as a subposet? Let us clarify what is meant by containment in this context. Given posets P and P', we say that P is *weakly contained* in P' if there exists an injective map $\psi : P \longrightarrow P'$ such that for any $x, y \in P$, $\psi(x) \leq_{P'} \psi(y)$ if $x \leq_P y$. We say that P' *strongly contains* P if in fact for any $x, y \in P$, $x \leq_P y$ if and only if $\psi(x) \leq_{P'} \psi(y)$. We also say in this case that P' contains P as an *induced* poset. Given a poset P, $ex(n, P)$ is defined by

$$ex(n, P) = \max\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{P}[n] \text{ and } \mathcal{F} \text{ does not weakly contain } P\},$$

and $ex^*(n, P)$ is defined by

$$ex^*(n, P) = \max\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{P}[n] \text{ and } \mathcal{F} \text{ does not strongly contain } P\}.$$

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There has been a considerable amount of research devoted to determining the asymptotic behaviour of $ex(n,P)$ and $ex^*(n,P)$. The first result of this kind is Sperner's Theorem [12], which states that the maximal size of an antichain of $\mathcal{P}[n]$ is $\binom{n}{\lfloor n/2\rfloor}$. In other words, if P_k is defined to be the chain poset of length k, then Sperner's result says that $ex(n,P_2)=\binom{n}{\lfloor n/2\rfloor}$. This was later extended by Erdős [4], who proved that $ex(n,P_k)$ is the sum of the $k-1$ largest binomial coefficients of order n. Of course, in the case of chains, $ex(n,P_k)$ and $ex^*(n,P_k)$ are the same number. A systematic study of $ex(n,P)$ for various specific posets was undertaken by a number of authors. For example, in the case of the diamond poset D_2 (which is also the hypercube of dimension 2), the best bound to date is $ex(n,D_2) = O\left((2.25+o(1))\binom{n}{\lfloor n/2\rfloor}\right)$ (due to Kramer, Martin and Young [7]), but it is conjectured that $ex(n,D_2)=O\left((2+o(1))\binom{n}{\lfloor n/2\rfloor}\right)$. It is not known whether the limit $\lim_{n\to\infty} ex(n,P)/\binom{n}{\lfloor n/2\rfloor}$ or the limit $\lim_{n\to\infty} ex^*(n,P)/\binom{n}{\lfloor n/2\rfloor}$ exists for every P. If these numbers do exists, then we denote them by $\pi(P)$ and $\pi^*(P)$, respectively. A central conjecture in this field is the following.

Conjecture 1.1. *For each poset P, $\pi(P)$ exists and is equal to $e(P)$, where $e(P)$ is the maximal number m such that for all n, the m middle layers of $\mathcal{P}[n]$ do not weakly contain P.*

This conjecture is attributed by Griggs, Li and Lu [5] to Saks and Winkler, who made the (unpublished) observation that whenever $\pi(P)$ was known to exist it was also known to equal $e(P)$. The conjecture was proved to be true for posets whose Hasse diagram is a tree by Bukh [2].

In general, much less is known about induced containment than weak containment. Boehnlein and Jiang [1] extended Bukh's result [2] to the induced case, i.e. they showed that the induced equivalent of Conjecture 1.1 is also true for every tree poset. Carroll and Katona [3] proved that

$$\binom{n}{\lfloor n/2\rfloor}(1+1/n+\Omega(1/n^2)) \le ex^*(n,V_2) \le \binom{n}{\lfloor n/2\rfloor}(1+2/n+O(1/n^2)),$$

where V_2 denotes the poset on three elements a,b,c where b and c are incomparable and both larger than a.

It follows at once from the result of Erdős mentioned above that $ex(n,P)\le (|P|+1)\binom{n}{\lfloor n/2\rfloor}$. However it was unknown until recently whether for every fixed poset P there exists a constant $c(P)$ such that if $\mathcal{F}\subseteq\mathcal{P}[n]$ does not contain P as an induced subposet, then $|\mathcal{F}|\le c(P)\binom{n}{\lfloor n/2\rfloor}$. The existence of such a constant was conjectured to be true by Katona and by Lu and Milans [9], and was proved by Methuku and Pálvölgyi [11] by means of a generalization of the Marcus-Tardos theorem about forbidden permutation matrices in 0-1 matrices [10].

Theorem 1.2 (Methuku, Pálvölgyi [11]). *For every poset P there exists $c(P)$ such that if $\mathcal{F} \subseteq \mathcal{P}[n]$ for some $n \in \mathbb{N}$ and $\mathcal{F}$ does not contain P as an induced subposet, then $|\mathcal{F}| \leq c(P)\binom{n}{\lfloor n/2 \rfloor}$.*

Given $\mathcal{F} \subseteq \mathcal{P}[n]$, the *Lubell mass* of $\mathcal{F}$, denoted by $l(\mathcal{F})$, is the quantity

$$l(\mathcal{F}) = \sum_{F \in \mathcal{F}} 1/\binom{n}{|F|}.$$

Notice that the Lubell mass of $\mathcal{F}$ is the expected number of times a maximal chain in $\mathcal{P}[n]$ chosen uniformly at random meets $\mathcal{F}$. Based on this observation, Lubell [8] proved that if $\mathcal{F} \subseteq \mathcal{P}[n]$ is an antichain then $l(\mathcal{F}) \leq 1$, and this naturally implies Sperner's Theorem. More generally, Lubell's argument shows that if $\mathcal{F}$ contains no P_k then $l(\mathcal{F}) \leq k-1$. The Lubell mass is a powerful tool for studying both $ex(n, P)$ and $ex^*(n, P)$, and has been used in various papers; for example Griggs, Li and Lu [5] explicitly make use of it in order to bound $ex(n, D_2)$. Let us remark here that $l(\mathcal{F})$ is technically a function of both $\mathcal{F}$ and the ground set $[n]$. Thus, to be precise, we should write $l_{[n]}(\mathcal{F})$ instead of $l(\mathcal{F})$. However, unless otherwise stated, the ground set is assumed to be $[n]$ and so we shall often simply write $l(\mathcal{F})$.

We see that, since $l(\mathcal{F}) \leq k-1$ for every family $\mathcal{F}$ not containing P_k, then $l(\mathcal{F}) \leq |P|-1$ for every family not containing a fixed poset P weakly. The purpose of this paper is to prove a corresponding resut for induced containment.

Theorem 1.3. *For every poset P there exists $c(P)$ such that if $\mathcal{F} \subseteq \mathcal{P}[n]$ for some $n \in \mathbb{N}$ and $\mathcal{F}$ does not contain P as an induced subposet, then $l(\mathcal{F}) \leq c(P)$.*

Theorem 1.3, which strengthens Theorem 1.2, was conjectured to be true by Lu and Milans [9]. They proved it in a number of special cases including posets of height at most 2, "series-parallel" posets, and hypercubes of dimension at most 3. We refer the reader to their paper [9] for more details.

It is easily seen that Theorem 1.3 implies Theorem 1.2. In fact, the Lubell mass of a family $\mathcal{F}$ can be viewed as a weighting of the elements of the hypercube, so that each layer of the hypercube is given the same total weight. Thus one of the main advantages of Theorem 1.3 is that it gives information about the low (and high) layers of $\mathcal{F}$, whereas Theorem 1.2 does not.

The rest of this paper is organized as follows. In Section 2, we reduce the problem of finding an induced poset P to finding an element of certain families of "universal" posets. In Section 3, we generalize the notion of *pivots* which were introduced by Lu and Milans in [9]. Then in Section 4, we explain the basic idea behind the proof of Theorem 1.3, and this serves

to demonstrate the usefulness of generalizing the concept of pivots. Section 5 is devoted to the proof of a key technical fact, Lemma 5.3, which will subsequently allow us to prove Theorem 1.3 in Section 6.

2 Universal posets

Throughout this paper we shall use the following standard notation. Given $n, m \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$, and if X is a set then $X^{(m)}$ denotes the set $\{Y \subseteq X : |Y| = m\}$; the elements of $X^{(m)}$ will be referred to as the m-subsets of X. Also, $X^{(\leq m)}$ denotes $\bigcup_{i=0}^{m} X^{(i)}$. Given $a, b \in \mathbb{Z}$ with $a \leq b$, $[a, b]$ denotes the set $\{a, a+1, \ldots, b\}$.

Let $n \in \mathbb{N}$. Let Q_n^+ be the partially ordered set obtained by ordering $\mathcal{P}[n]$ so that $x \leq y$ if and only if $x \subseteq y$. Thus Q_n^+ is simply the hypercube with the usual partial order. Let Q_n^- be the partially ordered set obtained by ordering $\mathcal{P}[n]$ so that $x \leq y$ if and only if $x \supseteq y$. Obviously Q_n^+ and Q_n^- are isomorphic, but it will be convenient in our proofs to have access to both of them.

We begin with a simple lemma about partially ordered sets, observed in [9].

Lemma 2.1. *Let P be a partially ordered set. Then P is an induced subposet of $Q^+_{|P|}$.*

Proof. Identify the elements of $[|P|]$ with those of P, i.e. let $n_x : x \in P$ be an enumeration of $[|P|]$. Consider the map $\psi : P \longrightarrow [|P|]$ defined by $\psi(x) = \{n_z : z \in P, z \leq x\}$. Let us check that ψ is an injective map which respects the order relation on P. To do so, it suffices to prove that if $x, y \in P$ and $x < y$ then $\psi(x) \subsetneq \psi(y)$, and if x, y are incomparable then $\psi(x)$ and $\psi(y)$ are incomparable too. So let $x, y \in P$. If $x < y$ then $\psi(x) = \{n_z : z \in P, z \leq x\} \subseteq \{n_z : z \in P, z \leq y\} = \psi(y)$. Besides, $n_y \notin \psi(x)$ as $y \not\leq x$. Thus $\psi(x) \subsetneq \psi(y)$. If x, y are incomparable then since $x \not\leq y$, $n_x \notin \psi(y)$ and since $y \not\leq x$, $n_y \notin \psi(x)$. Hence $\psi(x)$ and $\psi(y)$ are incomparable. $\square$

By Lemma 2.1, in order to find an induced copy of a poset P it is enough to find an induced copy of $Q^+_{|P|}$. However, an important idea of the proof of Theorem 1.3, rather than trying to find a copy of $Q^+_{|P|}$ directly, is to find a copy of a "dense" subposet which itself contains the desired copy of $Q^+_{|P|}$. We formalize this idea in what follows.

Let $m, n \in \mathbb{N}$. We let $U(n, m)$ denote the partially ordered set obtained by ordering the elements of $[n]^{(\leq m)}$ so that $x \leq y$ if and only if $x \subseteq y$. Similarly $D(n, m)$ denotes the partially ordered set obtained by ordering the elements of $[n]^{(\leq m)}$ so that $x \leq y$ if and only if $x \supseteq y$.

Let $\epsilon > 0$. We let $\mathcal{U}(n, m, \epsilon)$ be the set of induced subposets U of $U(n, m)$ such that $|U \cap [n]^{(i)}| \geq (1 - \epsilon)\binom{n}{i}$ for every i, $0 \leq i \leq m$. $\mathcal{D}(n, m, \epsilon)$ is the set of induced subposets D of $D(n, m)$ such that $|D \cap [n]^{(i)}| \geq (1 - \epsilon)\binom{n}{i}$ for every i, $0 \leq i \leq m$.

Lemma 2.2. *Let $m \in \mathbb{N}$. There exists $\epsilon > 0$ such that for all $n \geq 2m$, every $U \in \mathcal{U}(n, m, \epsilon)$ contains an induced copy of Q_m^+, and every $D \in \mathcal{D}(n, m, \epsilon)$ contains an induced copy of Q_m^-.*

Proof. Let $\epsilon = 1/(2m)^{m+1}$. Let $n \geq 2m$. Let $p = 2m/n$. Let $U \in \mathcal{U}(n, m, \epsilon)$. Let $\overline{U} = \{x \in [n]^{(\leq m)} : x \notin U\}$. To prove the lemma, it is clearly sufficient to find a subset of $[n]$ of size m not containing any element of $\overline{U}$. Choose a random subset X of $[n]$, each element of $[n]$ being selected independently of the others with probability p. Let Z be the number of elements of $[n]$ selected, so $\mathrm{E}(Z) = 2m$, and for $0 \leq i \leq m$ let Z_i be the number of elements of $\overline{U} \cap [n]^{(i)}$ contained in X. By assumption, $|\overline{U} \cap [n]^{(i)}| \leq \epsilon\binom{n}{i}$, so $\mathrm{E}(Z_i) \leq \epsilon p^i \binom{n}{i} \leq 1/(2m)$ by our choice of ϵ and p. Note that in fact $Z_0 = 0$ since $\emptyset \in U$ (because it is the only element of size 0 and $\epsilon > 0$). Therefore $\mathrm{E}(Z - \sum_{0 \leq i \leq m} Z_i) \geq m$ and so there exists a set X for which $Z - \sum_{0 \leq i \leq m} Z_i \geq m$. After removing from X one vertex from every element of $\overline{U}$ contained in X we are left with a set X' of size at least m and which does not contain any element of $\overline{U}$. The statement about $D(n, m, \epsilon)$ follows by symmetry. $\square$

3 Generalized pivots

Let $n \in \mathbb{N}$ and let $\mathcal{F}$ be a family of subsets of $[n]$. Let $r \in \mathbb{N}$. An *r-pivot* of A is an element X of $A^{(r)}$ such that there exists an element Y of $([n] \backslash A)^{(r)}$ and an element B of $\mathcal{F}$ such that $B = (A \backslash X) \cup Y$. In other words, we can obtain a subset in $\mathcal{F}$ by deleting the r elements of X from A and replacing them by r new elements. We say that B is a *witness* of X (being a pivot of A). If $r = 0$ then we view $\emptyset$ as being a 0-pivot for any $A \in \mathcal{F}$, A itself being the witness that $\emptyset$ is a pivot of A. An *r-anti-pivot* is an element Y of $([n] \backslash A)^{(r)}$ such that there exists an element X of $A^{(r)}$ and an element B of $\mathcal{F}$ such that $B = (A \backslash X) \cup Y$. The concept of r-pivots generalizes that of pivots introduced in [9] by Lu and Milans. A simple but quite important observation about r-pivots is as follows.

Observation 3.1. *Let $A \subseteq [n]$ and let $X \in A^{(r)}$ be an r-pivot of A with witness B. Then for any $F \subseteq A$, $F \subseteq B$ if and only if $F \cap X = \emptyset$. Likewise, if $Y \in ([n] \backslash A)^{(r)}$ is an r-anti-pivot of A with witness B, then for any $F \supseteq A$, $B \subseteq F$ if and only if $Y \subseteq F$.*

We generalize another related concept from [9]. Let $\gamma \in (0, 1]$ be a real number. We say that a set $A \in \mathcal{F}$ is *(γ, r)-flexible* (in $\mathcal{F}$) if it has at least

$(1-\gamma)\binom{|A|}{r}$ r-pivots, and is (γ,r)-*anti-flexible* if it has at least $(1-\gamma)\binom{n-|A|}{r}$ r-anti-pivots.

An important ingredient of the proof of Theorem 1.3 in the case of posets of height two in [9] was that a family $\mathcal{F}$ containing no large set as well as no $(\gamma,1)$-flexible sets has bounded Lubell mass. We prove that the same is true for generalized pivots.

Lemma 3.2. *Let $r,n\in\mathbb{N}$, let $\gamma\in(0,1]$. Let $\mathcal{F}\subseteq\mathcal{P}[n]$ be a family not containing any (γ,r)-flexible set and not containing any set of size more than $n/2$. Then $l(\mathcal{F})\le f(\gamma,r)$, where $f(\gamma,r)=r+2r^2\gamma^{-1}$.*

Proof. Let $r\le k\le n/2$. Construct a bipartite graph with vertex classes $\mathcal{F}\cap[n]^{(k)}$ and $[n]^{(k-r)}$ by drawing an edge between $A\in\mathcal{F}\cap[n]^{(k)}$ and $C\in[n]^{(k-r)}$ if $C\subseteq A$ and $A\backslash C$ is not an r-pivot of A. Let G be the subgraph obtained by deleting the vertices of degree zero in $[n]^{(k-r)}$, and let U and V be the two vertex classes of G, so that $U=V(G)\cap[n]^{(k)}$ and $V=V(G)\cap[n]^{(k-r)}$. Let us count $e(U,V)$, the number of edges of this bipartite graph, in two different ways. First, for every $A\in U$, since A is not (γ,r)-flexible, A must be joined to at least $\gamma\binom{k}{r}$ elements of V. Thus, $e(U,V)\ge\gamma\binom{k}{r}|U|$. Now let $C\in V$, so there exists $A\in U$ such that CA is an edge of G. Then any element of U other than A containing C must intersect $A\backslash C$ (else $A\backslash C$ would be a pivot of A). Now for each $x\in A\backslash C$, there are at most $\binom{n-k+r-1}{r-1}$ such sets containing x, so C has degree at most $1+r\binom{n-k+r-1}{r-1}\le 2r\binom{n-k+r-1}{r-1}=2r^2\binom{n-k+r}{r}/(n-k+r)$ in G. Hence $e(U,V)\le 2r^2\binom{n}{k-r}\binom{n-k+r}{r}/(n-k+r)=2r^2\binom{n}{k}\binom{k}{r}/(n-k+r)$. Thus by combining the two bounds on $e(U,V)$, we have $|U|/\binom{n}{k}\le 2r^2/(\gamma(n-k+r))$. Therefore

$$
\begin{aligned}
l(\mathcal{F}) &= \sum_{k=0}^{n}\frac{|\mathcal{F}\cap[n]^{(k)}|}{\binom{n}{k}}\\
&\le r+\sum_{k=r}^{n/2}\frac{|\mathcal{F}\cap[n]^{(k)}|}{\binom{n}{k}}\\
&\le r+2r^2\gamma^{-1}\sum_{k=r}^{n/2}\frac{1}{n-k}\\
&\le r+2r^2\gamma^{-1}.
\end{aligned}
$$

□

4 A first attempt

In order to motivate the introduction of r-pivots, as well as to explain the basic idea behind the proof of Theorem 1.3, we shall give in this section a brief sketch of a proof of Theorem 1.3 under two simplifying (but false!) assumptions. To this end, we first introduce some notation as well as a basic

and important lemma found in Lu and Milans [9]. We include a proof of it in an effort to make this paper self-contained.

Let $n \in \mathbb{N}$ and let $\mathcal{F} \subseteq \mathcal{P}[n]$. Let $A, B \subseteq [n]$ with $B \subseteq A$. We let $l_{B,A}(\mathcal{F})$ denote the expected number of times a random full chain in the interval $[B, A]$ meets $\mathcal{F}$ (the interval $[B, A]$ of $\mathcal{P}[n]$ is the set $\{X \in \mathcal{P}[n] : B \subseteq X \subseteq A\}$). Also, we let $\mathcal{F}_{B,A}$ denote the family $\{F \backslash B : F \in \mathcal{F}, B \subseteq F \subseteq A\}$. Notice that $l_{B,A}(\mathcal{F}) = l_{\emptyset, A\backslash B}(\mathcal{F}_{B,A})$. In other words, $l_{B,A}(\mathcal{F})$ is the Lubell mass of $\mathcal{F}_{B,A}$ viewed as a family of subsets of $A\backslash B$.

Lemma 4.1 (Lu, Milans [9])**.** *There exist $A, B \in \mathcal{F}$ such that $l_{\emptyset,A}(\mathcal{F}) \geq l(\mathcal{F})$ and $l_{B,[n]}(\mathcal{F}) \geq l(\mathcal{F})$.*

Proof. Choose a maximal chain $\mathcal{C}$ in $\mathcal{P}[n]$ uniformly at random. Let Z be the number of times $\mathcal{C}$ meets $\mathcal{F}$, so that $\mathrm{E}(Z) = l(\mathcal{F})$. Given $A \in \mathcal{F}$, let E_A be the event that A is the largest element of $\mathcal{F}$ that $\mathcal{C}$ contains. Also let E_* be the event that $\mathcal{C}$ meets no elements of $\mathcal{F}$. We then have $\mathrm{E}(Z) = \mathrm{P}(E^*)\mathrm{E}(Z|E^*) + \sum_{A\in\mathcal{F}} \mathrm{P}(E_A)\mathrm{E}(Z|E_A)$. Now obviously $\mathrm{E}(Z|E^*) = 0$, and so $\mathrm{E}(Z) = \sum_{A\in\mathcal{F}} \mathrm{P}(E_A)\mathrm{E}(Z|E_A)$. But, clearly, $\mathrm{E}(Z|E_A) = l_{\emptyset,A}(\mathcal{F})$ for every $A \in \mathcal{F}$. Hence there must exist A such that $l_{\emptyset,A}(\mathcal{F}) \geq l(\mathcal{F})$. The second part of the statement follows by symmetry. $\square$

Now that we have stated Lemma 4.1, we are ready to give our incorrect proof of Theorem 1.3. Let $n \in \mathbb{N}$ and let $\mathcal{F} \subseteq \mathcal{P}[n]$. Let $\mathcal{F}_+ = \{F \in \mathcal{F} : |F| \geq n/2$ and let $\mathcal{F}_- = \{F \in \mathcal{F} : |F| \leq n/2\}$. Our first assumption shall be that whatever $\mathcal{F}$ is, it is always true that $l(\mathcal{F}_-) \geq l(\mathcal{F})/2$. Our second assumption shall be that Lemma 3.2 is also true if $\gamma = 0$, i.e. there exists some constant b_r depending on r only such that $l(\mathcal{F}) \leq b_r$ for every family $\mathcal{F}$ containing no set A for which each element of $A^{(r)}$ is a pivot, as well as no set of size more than $n/2$.

Suppose then that P is some arbitrary poset. Let $m = |P|$. Suppose that $l(\mathcal{F}) \geq 2b_1 + 4b_2 + \cdots + 2^m b_m + 2^{m+1}(m+1)$. We begin by finding sequences $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_m$ and $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ such that for each $i \in [0, m]$, $\mathcal{F}_i \subseteq \mathcal{F}$, $F \subseteq A_i$ for each $F \in \mathcal{F}_i$, and finally A_i is a $(0, i)$-flexible set in $\mathcal{F}_{i-1}$. Moreover, we make sure that $l_{\emptyset,A_m}(\mathcal{F}_m) \geq m + 1$. To do so, note by Lemma 4.1 that there exists $A_0 \in \mathcal{F}$ such that $l_{\emptyset,A_0}(\mathcal{F}) \geq l(\mathcal{F})$. Let $\mathcal{F}_0 = \mathcal{F}_{\emptyset,A_0}$, so that $l(\mathcal{F}_0) \geq l(\mathcal{F})$ when viewed as a family of subsets of A_0. By our first assumption, $l_{\emptyset,A_0}((\mathcal{F}_0)_-) \geq l(\mathcal{F}_0)/2 = b_1 + 2b_2 + \cdots + 2^{m-1}b_m + 2^m(m + 1)$. Now let $(\mathcal{F}_0)^*_-$ be the set of elements of $\mathcal{F}$ which are not $(0, 1)$-flexible in $(\mathcal{F}_0)_-$. Notice that a pivot of A in $(\mathcal{F}_0)^*_-$ is certainly also a pivot of A in $(\mathcal{F}_0)_-$. Therefore the elements of $(\mathcal{F}_0)^*_-$ are not $(0, 1)$-flexible in $(\mathcal{F}_0)^*_-$ either. Hence by our second assumption, $l_{\emptyset,A_0}((\mathcal{F}_0)^*_-) \leq b_1$. Therefore by considering $(\mathcal{F}_0)_-\backslash(\mathcal{F}_0)^*_-$ we see that by Lemma 4.1, there exists $A_1 \in (\mathcal{F}_0)_-$ which is $(0, 1)$-flexible in $(\mathcal{F}_0)_-$ and moreover $l_{\emptyset,A_1}(\mathcal{F}_0) \geq l_{\emptyset,A_1}((\mathcal{F}_0)_-) - b_1 \geq l(\mathcal{F}_0)/2 - b_1 \geq b_2 + 2b_3 + \cdots + 2^{m-2}b_m + 2^{m-1}(m+1)$. We let $\mathcal{F}_1 = (\mathcal{F}_0)_{\emptyset,A_1}$, and iterate this procedure, now finding a set $A_2 \in \mathcal{F}_1$ which is $(0, 2)$-flexible

and such that $l_{\emptyset,A_2}(\mathcal{F}_1) \geq b_3 + 2b_4 + \cdots + 2^{m-3}b_m + 2^{m-2}(m+1)$, and so on. Iterating this procedure $m+1$ times yields the desired sequences.

Let $X = A_m$. Since $l_{\emptyset,A_m}(\mathcal{F}_m) \geq m+1$, X has size at least m. For each i, $0 \leq i \leq m$, and each $x \in X^{(i)}$, since x is an i-pivot of A_i in $\mathcal{F}_{i-1}$, there exists a witness $w_x \in \mathcal{F}_{i-1}$ for the fact that x is an i-pivot of A_i. Let $W = \bigcup_{x \in X^{(\leq m)}} \{w_x\}$. Notice that W is a subset of $\mathcal{F}$. Let us order $X^{(\leq m)}$ by reverse inclusion, so that that X, as a partially ordered set, is isomorphic to $D(|X|, m)$. Consider the injective map $\psi : X^{(\leq m)} \longrightarrow W$ which sends $x \in X^{(\leq m)}$ to its witness w_x. We claim that ψ preserves the order relation on $X^{(\leq m)}$. Indeed, suppose that $x, y \in X^{(\leq m)}$ where $w_x \in \mathcal{F}_{i-1}$ for some i and $w_y \in \mathcal{F}_{j-1}$ for some j and without loss of generality $i \leq j$. If $i = j$ and $x \neq y$, then x and y are incomparable in $X^{(\leq m)}$. It is also easy to see that w_x and w_y are two distinct elements of $\mathcal{F}$ of the same size and hence are incomparable in $\mathcal{F}$ too. So let us assume that $i < j$ (and so $x \neq y$). Clearly since $|x| < |y|$, either $x > y$ (if $x \subsetneq y$) or x and y are incomparable. If $x \subseteq y$, then by Observation 3.1, $w_y \subsetneq w_x$ since $w_y \subseteq A_i$. If x and y are incomparable in $X^{(\leq m)}$ then there exists $z \in x \backslash y$, but then since $x \subseteq X \subseteq A_j$, it must be the case that $z \in w_y$, and so since $z \notin w_x$, w_x and w_y must be incomparable. This proves that ψ preserves the order relation, and hence that $\mathcal{F}$ contains an induced copy of $D(|X|, m)$, hence one of Q_m^- (as $|X| \geq m$), and so one of P too.

Of the two assumptions that we made, the first one is the one which is most easily dealt with. Indeed, it is clear that for any $\mathcal{F} \subseteq \mathcal{P}[n]$, either $l(\mathcal{F}_-) \geq l(\mathcal{F})/2$ or $l(\mathcal{F}_+) \geq l(\mathcal{F})/2$. Therefore, in the proof of Theorem 1.3, we shall resort to a "two passes" argument; there will be $2m+1$ steps (rather than $m+1$) and for at least $m+1$ steps we shall know that the same alternative held (say $l(\mathcal{F}_-) \geq l(\mathcal{F})/2$), which, as it turns out, shall be enough to find a copy of P.

The second assumption is more difficult to overcome. Indeed, it is not possible to force all r-subsets of some element of $\mathcal{F}$ to be r-pivots. However, it is possible, by Lemma 3.2, to force a *very large* proportion of them to be r-pivots. How can one then make use of this? The key idea is that, having ensured that a very large proportion of the r-subsets of a set A are r-pivots, we can find a large subfamily of $\mathcal{F}_{\emptyset,A}$ such that each member F of this family is such that a very large proportion of its r-subsets are pivots of A. This will allow us to carry out the iterative procedure described above. In the next section, we shall make this idea more precise.

5 Families containing no fat sets

Let us begin with an important definition. Let $r, n \in \mathbb{N}$. Let $\mathcal{S} \subseteq [n]^{(r)}$. Let $X \subseteq [n]$. Let $\epsilon > 0$. We say that X is *$(\epsilon, \mathcal{S})$-fat* if $|X^{(r)} \cap \mathcal{S}| \geq (1-\epsilon)\binom{|X|}{r}$.

Our goal in this section is to prove Lemma 5.3, which says that a family

$\mathcal{F}$ containing no $(\epsilon, \mathcal{S})$-fat set has bounded Lubell mass provided $\mathcal{S}$ is large enough. Essentially all the work required to prove Lemma 5.3 is contained in Lemma 5.2, where we use the language of probability to bound the proportion of elements of a fixed layer of the hypercube which contain too many elements of a small subset $\mathcal{T}$ of $[n]^{(r)}$.

Let m, k, n be positive integers with $\max(m, k) \leq n$. The *hypergeometric distribution* with parameters m, k, n is the probability distribution of the random variable $Z = |X \cap [k]|$ where X is chosen uniformly at random in $[n]^{(m)}$. In other words, Z counts the number of elements which lie in $[k]$ of an m-subset of $[n]$ chosen uniformly at random. By Theorem 2.10 of [6], the following standard concentration inequality holds.

Lemma 5.1. *Let m, k, n be positive integers with $\max(m, k) \leq n$, and let Z be a hypergeometric random variable with parameters m, k, n. Let $t \geq 0$. Then*

$$\mathrm{P}\left(Z \geq \mathrm{E}(Z) + t\right) \leq \exp(-2t^2/m).$$

Lemma 5.1 enables us to prove the intuitively true fact if $\mathcal{T} \subseteq [n]^{(r)}$ is small then with very high probability a uniformly random m-subset of $[n]$ contains few elements of $\mathcal{T}$.

Lemma 5.2. *Let $\epsilon \in (0, 1]$. Let $r \in \mathbb{N}$. There exists $c(\epsilon, r) > 0$, $\eta(\epsilon, r) > 0$ and $m_0(\epsilon, r)$ such that, for each $n \geq m \geq m_0(\epsilon, r)$, if $\mathcal{T} \subseteq [n]^{(r)}$ satisfies $|\mathcal{T}| \leq \eta(\epsilon, r)\binom{n}{r}$, then*

$$\mathrm{P}\left(|X^{(r)} \cap \mathcal{T}| > \epsilon\binom{m}{r}\right) \leq \exp(-c(\epsilon, r)m),$$

where X is chosen uniformly at random in $[n]^{(m)}$.

Proof. It will be convenient here to view $\mathcal{T}$ as an r-uniform hypergraph with vertex set $[n]$. In the light of this, let us remind the reader of two standard definitions. Let $v \in [n]$. The *link graph* of v in X is the set $L_X(v) = \{e\backslash\{v\} : e \in \mathcal{T},\ v \in e \text{ and } e \subseteq X\}$. The *degree* of v in X, denoted by $d_X(v)$, is defined to be $|L_X(v)|$. When $X = [n]$ we simply write $L(v)$ and $d(v)$.

For $r = 0$, we can set $\eta(\epsilon, 0) = 1/2$, $c(\epsilon, 0) = 1$ and $m_0(\epsilon, 0) = 0$ for any ϵ (in fact, any $\eta(\epsilon, 0) < 1$ and any $c(\epsilon, 0) > 0$ would do). So let us focus on the case $r \geq 1$. We shall prove the lemma by induction on r. First consider the base case $r = 1$. Set $\eta(\epsilon, 1) = \epsilon/2$ and $m_0(\epsilon, 1) = 1$ and suppose $\mathcal{T} \subseteq [n]$ has size at most $\eta(\epsilon, 1)n$. Let $m \geq m_0(\epsilon, 1)$, and choose $X \in [n]^{(m)}$ uniformly at random. Then $|X \cap \mathcal{T}|$ is hypergeometrically distributed with parameters m, $k = |\mathcal{T}|$, n, and $\mathrm{E}(|X \cap \mathcal{T}|) = km/n \leq \epsilon m/2$. Therefore, by Lemma 5.1,

$$\begin{aligned} \mathrm{P}(|X \cap \mathcal{T}| > \epsilon m) &= \mathrm{P}(|X \cap \mathcal{T}| > \epsilon m/2 + \epsilon m/2) \\ &\leq \mathrm{P}(|X \cap \mathcal{T}| \geq \mathrm{E}(|X \cap \mathcal{T}|) + \epsilon m/2) \\ &\leq \exp(-\epsilon^2 m/2). \end{aligned}$$

Thus the base case holds if we set $c(\epsilon, 1) = \epsilon^2/2$.

Suppose now that the induction hypothesis holds for all r' with $1 \leq r' \leq r-1$, and let us prove that it holds for r. In other words, we assume that values of c, η and m_0 exist and satisfy the requirements of the lemma for any $\epsilon > 0$ and $r' \in [1, r-1]$. Let $\delta_1 = \eta(\epsilon/2, r-1)$, let $\delta_2 = \eta(\epsilon/2, 1)$ and set $\eta(\epsilon, r) = \delta_1\delta_2$. Let $\mathcal{T} \subseteq [n]^{(r)}$ with $|\mathcal{T}| \leq \eta(\epsilon, r)\binom{n}{r}$. Let $c(\epsilon, r) = \min\{c(\epsilon/2, 1), c(\epsilon/2, r-1)\}/2$. Let m_* be sufficiently large that

$$\exp(-c(\epsilon/2, 1)m) + m\exp(-c(\epsilon/2, r-1)(m-1)) \leq \exp(-c(\epsilon, r)m)$$

for all $m \geq m_*$. Let $m_0(\epsilon, r) = \max\{m_0(\epsilon/2, 1), m_0(\epsilon/2, r-1) + 1, m_*\}$. Choose $X \in [n]^{(m)}$ uniformly at random.

Let us bound the probability that $|X^{(r)} \cap \mathcal{T}| > \epsilon\binom{m}{r}$. We define two sets V_1 and V_2 of elements of $[n]$ as follows. Let $V_1 = \{v \in [n] : d(v) > \delta_1\binom{n-1}{r-1}\}$ and let $V_2 = \{v \in [n] : d(v) \leq \delta_1\binom{n-1}{r-1}\}$. V_1 and V_2 partition $[n]$ and we view the elements of V_1 as having high degree and those of V_2 as having low degree.

Let A be the event that $|X \cap V_1| > \epsilon m/2$. We wish to bound the probability that A occurs. For this, we first find an upper bound on $|V_1|$. Each vertex of V_1 contains at least $\delta_1\binom{n-1}{r-1}$ elements of $\mathcal{T}$, so $r|\mathcal{T}| \geq \delta_1\binom{n-1}{r-1}|V_1|$, hence

$$\begin{aligned}
|V_1| &\leq \frac{r|\mathcal{T}|}{\delta_1\binom{n-1}{r-1}} \\
&\leq \frac{r\eta(\epsilon, r)\binom{n}{r}}{\delta_1\binom{n-1}{r-1}} \\
&= \eta(\epsilon, r)n/\delta_1 \\
&= \delta_2 n.
\end{aligned}$$

By the choice of δ_2 and since $m \geq m_0(\epsilon/2, 1)$, we then have $\mathrm{P}(A) \leq \exp(-c(\epsilon/2, 1)m)$.

For $v \in V_2$, let B_v be the event that $|X^{(r-1)} \cap L(v)| > \epsilon\binom{m-1}{r-1}/2$, conditional on $v \in X$. Fix $v \in V_2$. Let us bound the probability of B_v. Given that $v \in X$, $X\backslash\{v\}$ is a uniformly random subset of $[n]\backslash\{v\}$ of size $m-1$. Since $|L(v)| \leq \delta_1\binom{n-1}{r-1}$ and $\delta_1 = \eta(\epsilon/2, r-1)$, and since $m - 1 \geq m_0(\epsilon/2, r-1)$, the probability that B_v occurs is no more than $\exp(-c(\epsilon/2, r-1)(m-1))$. Now let B be the event that there is some element v of V_2 which belongs to X and for which $d_X(v) > \epsilon\binom{m-1}{r-1}/2$. By a simple union bound,

$$\begin{aligned}
\mathrm{P}(B) &\leq \sum_{v \in V_2} \mathrm{P}(v \in X)\mathrm{P}(B_v) \\
&\leq |V_2|(m/n)\exp(-c(\epsilon/2, r-1)(m-1)) \\
&\leq m\exp(-c(\epsilon/2, r-1)(m-1)).
\end{aligned}$$

If neither A nor B occurs, then

$$\begin{aligned}
r|X^{(r)} \cap \mathcal{T}| &= \sum_{v \in X \cap V_1} d_X(v) + \sum_{v \in X \cap V_2} d_X(v) \\
&\leq |X \cap V_1| \binom{m-1}{r-1} + m \max_{v \in X \cap V_2} d_X(v) \\
&\leq (\epsilon m/2) \binom{m-1}{r-1} + (\epsilon m/2) \binom{m-1}{r-1} \\
&= \epsilon m \binom{m-1}{r-1}.
\end{aligned}$$

Thus $|X^{(r)} \cap \mathcal{T}| \leq \epsilon \binom{m}{r}$. This implies that

$$\begin{aligned}
\mathrm{P}\left(|X^{(r)} \cap \mathcal{T}| > \epsilon \binom{m}{r} \right) &\leq \mathrm{P}(A) + \mathrm{P}(B) \\
&\leq \exp(-c(\epsilon/2, 1)m) + m \exp(-c(\epsilon/2, r-1)(m-1)) \\
&\leq \exp(-c(\epsilon, r)m),
\end{aligned}$$

the last inequality holding since $m \geq m_*$. $\square$

Lemma 5.3. *Let $r \in \mathbb{N}$. Let $\epsilon > 0$. There exists $h(\epsilon, r)$ such that if $\mathcal{S} \subseteq [n]^{(r)}$ and $|\mathcal{S}| \geq (1 - \eta(\epsilon, h))\binom{n}{r}$ then $l(\mathcal{F}) \leq h(\epsilon, r)$ for every $\mathcal{F} \subseteq \mathcal{P}[n]$ which contains no $(\epsilon, \mathcal{S})$-fat set, where $\eta(\epsilon, r)$ is the constant defined in Lemma 5.2 and h is a constant depending on ϵ and r only.*

Proof. If $\mathcal{F} \subseteq \mathcal{P}[n]$ contains no $(\epsilon, \mathcal{S})$-fat set then $|F^{(r)} \cap \overline{\mathcal{S}}| > \epsilon \binom{m}{r}$ for every $F \in \mathcal{F}$, where $\overline{\mathcal{S}}$ is the complement of $\mathcal{S}$ in $[n]^{(r)}$, i.e. $\overline{\mathcal{S}} = \{x \in [n]^{(r)} : x \notin \mathcal{S}\}$. By assumption $|\overline{\mathcal{S}}| \leq \eta(\epsilon, r)\binom{n}{r}$, hence by Lemma 5.2 applied to $\mathcal{T} = \overline{\mathcal{S}}$, we have

$$\begin{aligned}
l(\mathcal{F}) &= \sum_{m=0}^{n} \frac{|\mathcal{F} \cap [n]^{(m)}|}{\binom{n}{m}} \\
&\leq m_0(\epsilon, r) + \sum_{m=m_0(\epsilon,r)}^{n} \exp(-c(\epsilon, r)m) \\
&\leq m_0(\epsilon, r) + 1/(1 - e^{-c(\epsilon,r)}).
\end{aligned}$$

Thus the lemma certainly holds if we let $h(\epsilon, r) = m_0(\epsilon, r) + 1/(1 - e^{-c(\epsilon,r)})$. $\square$

6 Proof of Theorem 1.3

We are now ready to prove Theorem 1.3. The backbone of the proof is the same as the argument described in Section 4. We stress this point because

it is easy to get lost in the details of the proof below. Lemma 6.1 represents one step of the iteration, while Lemma 6.2 is in essence the result of applying Lemma 6.1 (at most) $2m+1$ times. After proving these two lemmas, we finish the proof of the theorem by extracting the desired poset from the sequences that we constructed.

Let P be a partially ordered set. Let $m = |P|$. By Lemmas 2.1 and 2.2 there exists $\epsilon > 0$ such that for each $n \geq 2m$, every $U \in \mathcal{U}(n, m, \epsilon)$ and every $D \in \mathcal{D}(n, m, \epsilon)$ contains an induced copy of P.

Let us define some constants. Here f represents the contant defined in Lemma 3.2, η represents the constant defined in Lemma 5.2 and h represents the constant defined in Lemma 5.3. Let

$$\epsilon_1 = \epsilon$$

and for $2 \leq j \leq 2m+1$ let

$$\epsilon_j = \min\{\epsilon_{j-1},\ \eta(\epsilon_{j-1}, i) : i \in [0, m]\}.$$

For $2 \leq j \leq 2m+1$, let

$$q_j = \max_{i \in [0,m]} h(\epsilon_{j-1}, i).$$

Let

$$q = \max_{j \in [2, 2m+1]} q_j.$$

Let

$$p = \max_{i \in [0,m],\, j \in [1, 2m+1]} f(\epsilon_j, i).$$

The constants ϵ_j, q and p above are defined precisely so that the following statement is true.

Lemma 6.1. *Let $0 \leq d \leq 2m$. Let $0 \leq a, b \leq m$. Let $n \in \mathbb{N}$. If $d \geq 1$, let $\mathcal{S}_0 \subseteq [n]^{(r_0)}$, $\mathcal{S}_1 \subseteq [n]^{(r_1)}, \ldots, \mathcal{S}_{d-1} \subseteq [n]^{(r_{d-1})}$ for some $0 \leq r_0, r_1, \ldots, r_{d-1} \leq m$, and suppose that $\mathcal{S}_i$ is $(\epsilon_{2m+2-d}, [n]^{(r_i)})$-fat for all i, $0 \leq i \leq d-1$. Let $\mathcal{F} \subseteq \mathcal{P}[n]$ with $l(\mathcal{F}) > 4mq + 2p$. Then there exists $Y \in \mathcal{F}$ such that*

1. *Either Y is (ϵ_{2m+1-d}, a)-flexible, $l_{\emptyset, Y}(\mathcal{F}) \geq l(\mathcal{F})/2 - 2mq - p$ and (if $d \geq 1$) Y is $(\epsilon_{2m+1-d}, \mathcal{S}_i)$-fat for all i, $0 \leq i \leq d-1$, or*
2. *Y is (ϵ_{2m+1-d}, b)-anti-flexible, $l_{Y,[n]}(\mathcal{F}) \geq l(\mathcal{F})/2 - 2mq - p$ and (if $d \geq 1$) $[n] \backslash Y$ is $(\epsilon_{2m+1-d}, \mathcal{S}_i)$-fat for all i, $0 \leq i \leq d-1$.*

Proof. Let $\mathcal{F}_- = \{F \in \mathcal{F} : |F| \leq n/2\}$ and $\mathcal{F}_+ = \{F \in \mathcal{F} : |F| \geq n/2\}$. Clearly either $l(\mathcal{F}_-) \geq l(\mathcal{F})/2$ or $l(\mathcal{F}_+) \geq l(\mathcal{F})/2$.

Suppose first that $l(\mathcal{F}_-) \geq l(\mathcal{F})/2$. Let $\mathcal{F}_-^*$ be the elements of $\mathcal{F}_-$ which are not (ϵ_{2m+1-d}, a)-flexible. By Lemma 3.2, we have $l(\mathcal{F}_-^*) \leq f(\epsilon_{2m+1-d}, a) \leq$

p (here notice that if F is not (ϵ_{2m+1-d}, a)-flexible in $\mathcal{F}_-$ then it isn't (ϵ_{2m+1-d}, a)-flexible in $\mathcal{F}_-^*$ either). If $d \geq 1$, for each $i \in [0, d-1]$ let $\mathcal{F}_-^i$ be the elements of $\mathcal{F}_-$ which are not $(\mathcal{S}_i, \epsilon_{2m+1-d})$-fat; by definition of ϵ_{2m+2-d}, we have $\epsilon_{2m+2-d} \leq \eta(\epsilon_{2m+1-d}, r_i)$, and by definition of q, we have $q \geq q_{2m+2-d} \geq h(\epsilon_{2m+1-d}, r_i)$ so that, by Lemma 5.3, $l(\mathcal{F}_-^i) \leq q$. Now if $d = 0$ let $\mathcal{F}_-^\star = \mathcal{F}_- \backslash \mathcal{F}_-^*$, and if $d \geq 1$ let $\mathcal{F}_-^\star = \mathcal{F}_- \backslash \left(\mathcal{F}_-^* \cup \mathcal{F}_-^0 \cup \mathcal{F}_-^1 \cup \mathcal{F}_-^2 \cup \cdots \cup \mathcal{F}_-^{d-1} \right)$. Either way, we clearly have $l(\mathcal{F}_-^\star) \geq l(\mathcal{F}_-) - 2mq - p \geq l(\mathcal{F})/2 - 2mq - p > 0$. By Lemma 4.1 there exists $Y \in \mathcal{F}_-^\star$ with $l_{\emptyset,Y}(\mathcal{F}_-^\star) \geq l(\mathcal{F}_-^\star) \geq l(\mathcal{F})/2 - 2mq - p$. Y satisifies the requirements of the lemma in this case.

Suppose now that $l(\mathcal{F}_+) \geq l(\mathcal{F})/2$. Let $\widetilde{\mathcal{F}} = \{[n]\backslash F : F \in \mathcal{F}_+\}$. Then $|F| \leq n/2$ for every $F \in \widetilde{\mathcal{F}}$ and $l(\widetilde{\mathcal{F}}) = l(\mathcal{F}_+)$. Therefore by the same argument as in the previous case, there exists $Y \in \widetilde{\mathcal{F}}$ which is $(\epsilon_{2m+1-d}, \mathcal{S}_i)$-fat for all i, $0 \leq i \leq d-1$ (if $d \geq 1$), is (b, ϵ_{2m+1-d})-flexible and $l_{\emptyset,Y}(\mathcal{F}) \geq l(\widetilde{\mathcal{F}}) - 2mq - p$. But notice that a b-pivot for Y corresponds to a b-anti-pivot of $[n]\backslash Y \in \mathcal{F}_+$, so that if we let $Y' = [n]\backslash Y$ then Y' is (ϵ_{2m+1-d}, b)-anti-flexible. Moreover $l_{\emptyset,Y}(\widetilde{\mathcal{F}}) = l_{[n]\backslash Y,[n]}(\mathcal{F}_+)$. Thus Y' satisfies the requirements of the lemma in this case. □

Let us remark that, while we stated Lemma 6.1 for a family of subsets of $[n]$ for ease of notation, the lemma obviously remains true for a family of subsets of any set Z; in fact, in the next lemma, we shall apply Lemma 6.1 to families whose ground sets are subsets of $[n]$.

Lemma 6.2. *Let $\mathcal{F} \subseteq \mathcal{P}[n]$ with*

$$l(\mathcal{F}) \geq 2^{2m+1}(2m+1) + \sum_{i=1}^{2m+1} 2^i(4mq + 2p).$$

Then there exists $t \in [0, 2m]$ and sequences $(\mathcal{F}_i)_{i=-1}^t$, $(A_i)_{i=-1}^t$, $(B_i)_{i=-1}^t$, $(a_i)_{i=-1}^t$, $(b_i)_{i=-1}^t$, $(\mathcal{S}_i)_{i=0}^t$ where $\mathcal{F}_{-1} = \mathcal{F}$, $A_{-1} = [n]$, $B_{-1} = \emptyset$, $a_{-1} = b_{-1} = -1$, and

1. *For each $i \in [-1, t]$, $\mathcal{F}_i$ is a subfamily of $\mathcal{F}$ with $B_i \subseteq F \subseteq A_i$ for all $F \in \mathcal{F}$;*

2. *For each $i \in [0, t]$, either $a_i = a_{i-1} + 1$ and $b_i = b_{i-1}$, or $a_i = a_{i-1}$ and $b_i = b_{i-1} + 1$;*

3. *For each $i \in [0, t]$, if $a_i = a_{i-1} + 1$ then $A_i \in \mathcal{F}_{i-1}$, $B_i = B_{i-1}$ and $A_i \backslash B_{i-1}$ is an (ϵ_{2m+1-t}, a_i)-flexible set in $(\mathcal{F}_{i-1})_{B_{i-1},A_{i-1}}$ with set of a_i-pivots $\mathcal{S}_i$. If on the other hand $b_i = b_{i+1} + 1$ then $B_i \in \mathcal{F}_{i-1}$, $A_i = A_{i-1}$ and $B_i \backslash B_{i-1}$ is an (ϵ_{2m+1-t}, b_i)-anti-flexible set in $(\mathcal{F}_{i-1})_{B_{i-1},A_{i-1}}$ with set of b_i-anti-pivots $\mathcal{S}_i$;*

4. *$A_t \backslash B_t$ is $(\epsilon_{2m+1-t}, \mathcal{S}_i)$-fat for each $i \in [0, t]$;*

5. $l_{B_t,A_t}(\mathcal{F}_t) \geq 2^{2m-t}(2m+1) + \sum_{i=1}^{2m-t} 2^i(2mq+p)$;

6. *Either* $a_t = m$ *or* $b_t = m$.

Proof. We shall prove that if for some $d \in [0, 2m]$, there exist sequences $(\mathcal{F}_i)_{i=-1}^{d-1}$, $(A_i)_{i=-1}^{d-1}$, $(B_i)_{i=-1}^{d-1}$, $(a_i)_{i=-1}^{d-1}$, $(b_i)_{i=-1}^{d-1}$, $(\mathcal{S}_i)_{i=0}^{d-1}$ as above satisfying conditions 1 to 5 of the lemma except condition 6, then we can find $\mathcal{F}_d$, A_d, B_d, a_d, b_d and $\mathcal{S}_d$ to enlarge each sequence, so that the new sequences satisfy conditions 1 to 5 of the lemma and either $a_d = a_{d-1}+1$ or $b_d = b_{d-1}+1$. This is enough to prove the lemma. Indeed, we start with $a_{-1} = -1$ and $b_{-1} = -1$ and whenever we enlarge the sequences, either a_d increases by one or b_d increases by one. Hence after at most $2m+1$ enlargements, it must be the case that condition 6 of the lemma is satisfied, and we let t be the last value of d obtained.

So suppose that for some $d \in [0, 2m]$ we have sequences $(\mathcal{F}_i)_{i=-1}^{d-1}$, $(A_i)_{i=-1}^{d-1}$, $(B_i)_{i=-1}^{d-1}$, $(a_i)_{i=-1}^{d-1}$, $(b_i)_{i=-1}^{d-1}$, $(\mathcal{S}_i)_{i=0}^{d-1}$ satisfying conditions 1 to 5 of the lemma, but where $a_{d-1} < m$ and $b_{d-1} < m$. By condition 5, $l_{A_{d-1},B_{d-1}}(\mathcal{F}_{d-1}) \geq 2(2m+1) + 2(2mq+p) > 4mq + 2p$. Therefore we may apply Lemma 6.1 to $(\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}$, viewed as a family of subsets of $A_{d-1} \backslash B_{d-1}$. As we view the ground set of $(\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}$ to be $A_{d-1} \backslash B_{d-1}$, when applying the lemma, we take the sequence of $\mathcal{S}_i$'s to be $(\mathcal{S}'_i)_{i=0}^{d-1}$, where $\mathcal{S}'_i = \mathcal{S}_i \cap \mathcal{P}(A_{d-1} \backslash B_{d-1})$ for each i. We also let $a = a_{d-1}+1$ and $b = b_{d-1}+1$. There are two possible outcomes from applying the lemma.

Case 1. We obtain $Y \in (\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}$ which is $(\epsilon_{2m+1-d}, \mathcal{S}'_i)$-fat for all i, $0 \leq i \leq d-1$, Y is (ϵ_{2m+1-d}, a)-flexible and $l_{\emptyset,Y}((\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}) \geq l((\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}})/2 - 2mq - p$. We let $b_d = b_{d-1}$, $a_d = a$, $A_d = B_{d-1} \cup Y$, $B_d = B_{d-1}$,

$$\mathcal{F}_d = \{F \in \mathcal{F}_{d-1} : B_d \subseteq F \subseteq A_d\},$$

and we let $\mathcal{S}_d$ be the set of a-pivots for Y in $(\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}$.

Let us check that this is a valid enlargement of the sequences, i.e. that conditions 1 to 5 of the lemma are satisfied. Conditions 1 and 2 need only be checked for $i = d$ since for smaller i they are inherited from the sequences that we are enlarging. Condition 1 is immediately satisfied by the definition of $\mathcal{F}_d$. Condition 2 is equally trivally satisfied since $a_d = a = a_{d-1}+1$ and $b_d = b_{d-1}$. Condition 3 also only needs to be checked for $i = d$ because we know it holds for the sequences we are enlarging, and $\epsilon_{2m+1-(d-1)} \leq \epsilon_{2m+1-d}$. Now $Y \in (\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}$, so that $A_d = B_{d-1} \cup Y \in \mathcal{F}_{d-1}$, and $A_d \backslash B_{d-1} = Y$, which is (ϵ_{2m+1-d}, a)-flexible in $(\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}$, with sets of a-pivots $\mathcal{S}_d$. So condition 3 is satisfied. Condition 4 is satisfied because $A_d \backslash B_d = Y$, which is $(\epsilon_{2m+1-d}, \mathcal{S}'_i)$-fat for all $i \in [0, d-1]$ (hence $(\epsilon_{2m+1-d}, \mathcal{S}_i)$-fat too), and also $(\epsilon_{2m+1-d}, \mathcal{S}_d)$-fat by

definition of $\mathcal{S}_d$. Finally,

$$\begin{aligned} l_{B_d,A_d}(\mathcal{F}_d) &= l_{\emptyset,A_d\backslash B_d}((\mathcal{F}_d)_{B_d,A_d}) \\ &= l_{\emptyset,Y}((\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}) \\ &\geq l_{B_{d-1},A_{d-1}}(\mathcal{F}_{d-1})/2 - 2mq - p \\ &\geq 2^{2m-d}(2m+1) + \sum_{i=1}^{2m-d} 2^i(2mq+p), \end{aligned}$$

and so condition 5 is satisfied.

Case 2. We obtain $Y \in (\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}$ such that $(A_{d-1}\backslash B_{d-1})\backslash Y$ is $(\epsilon_{2m+1-d}, \mathcal{S}'_i)$-fat for all i, $0 \leq i \leq d-1$, Y is (ϵ_{2m+1-d}, b)-anti-flexible, and $l_{Y,B_{d-1}\backslash A_{d-1}}((\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}) \geq l((\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}})/2 - 2mq - p$. Then we let $a_d = a_{d-1}$, $b_d = b$, $A_d = A_{d-1}$, $B_d = B_{d-1} \cup Y$,

$$\mathcal{F}_d = \{F \in \mathcal{F}_{d-1} : B_d \subseteq F \subseteq A_d\},$$

and we let $\mathcal{S}_d$ be the set of b-anti-pivots for Y in $(\mathcal{F}_{d-1})_{B_{d-1},A_{d-1}}$.

It is again straightforward, but tedious, to check that this is a valid enlargement of the sequences. □

We are now able to finish the proof of Theorem 1.3. Let $n \in \mathbb{N}$ and let $\mathcal{F} \subseteq \mathcal{P}[n]$ with

$$l(\mathcal{F}) \geq 2^{2m+1}(2m+1) + \sum_{i=1}^{2m+1} 2^i(2mq+p).$$

Let $(\mathcal{F}_i)_{i=-1}^t$, $(A_i)_{i=-1}^t$, $(B_i)_{i=-1}^t$, $(a_i)_{i=-1}^t$, $(b_i)_{i=-1}^t$, $(\mathcal{S}_i)_{i=0}^t$ be the sequences satisfying the conclusion of Lemma 6.2. Notice that it follows from the definition of the sequences that $\emptyset = B_{-1} \subseteq B_0 \subseteq B_1 \subseteq \cdots \subseteq B_t \subseteq A_t \subseteq A_{t-1} \subseteq \cdots \subseteq A_0 \subseteq A_{-1} = [n]$.

There are two cases to consider: either $a_t = m$ or $b_t = m$. Suppose first that $a_t = m$. Then there exist $i_0 < i_1 < i_2 < \ldots < i_m$ such that for each k, $a_{i_k} = k$ and $a_{i_k} > a_{i_k-1}$, so that $\mathcal{S}_{i_k}$ is a set of k-pivots for $A_{i_k}\backslash B_{i_k-1}$ in $(\mathcal{F}_{i_k-1})_{B_{i_k-1},A_{i_k-1}}$. Notice that since all the elements of $(\mathcal{F}_{i_k-1})_{B_{i_k-1},A_{i_k-1}}$ are obtained from $\mathcal{F}_{i_k-1}$ by removing B_{i_k-1} from them and also $A_{i_k} \in \mathcal{F}_{i_k-1}$, $\mathcal{S}_{i_k}$ is also a set of k-pivots for A_{i_k} in $\mathcal{F}_{i_k-1}$.

Now, let $X = A_{i_m}\backslash B_{i_m}$. Since $l_{B_{i_m},A_{i_m}}(\mathcal{F}_{i_m}) \geq 2m+1$, $|X| \geq 2m$. By condition 5, X is $(\epsilon_{2m+1-i_m}, \mathcal{S}_{i_k})$-fat for all k, $0 \leq k \leq m$. As $\epsilon_{2m+1} \leq \epsilon_{2m} \leq \cdots \leq \epsilon_1 = \epsilon$, this implies that X is $(\epsilon, \mathcal{S}_{i_k})$-fat for each k, $0 \leq k \leq m$. For a fixed k, let $V_k = \mathcal{S}_{i_k} \cap X^{(k)}$, i.e. V_k is the set of k-pivots for A_{i_k} in $\mathcal{F}_{i_k-1}$ which are contained in X. For each $x \in V_k$, let w_x be a witness of x being a k-pivot of A_{i_k} in $\mathcal{F}_{i_k-1}$. Let $W_k = \bigcup_{x \in V_k}\{w_x\}$ and let $W = \bigcup_{k=0}^m W_k$. We shall prove the following claim.

Claim 6.3. *W, viewed as a subposet of $\mathcal{F}$, is isomorphic to an element of $\mathcal{D}(|X|, m, \epsilon)$.*

Proof. Order the elements of V so that $x \leq y$ if and only if $x \supseteq y$. Up to relabelling of the elements of X it is clear that $V \in \mathcal{D}(|X|, m, \epsilon)$, since X is (ϵ, V_k)-fat for every k, $0 \leq k \leq m$.

Let $\psi : V \longrightarrow W$ be the map sending $x \in V$ to $w_x \in W$. We shall prove that ψ is an isomorphism between $(V, \leq)$ and $(W, \subseteq)$, which proves the claim.

Suppose $x, y \in V$ where $w_x \in \mathcal{F}_{i_{k_1}-1}$ for some i_{k_1} and $w_y \in \mathcal{F}_{i_{k_2}-1}$ for some i_{k_2} and without loss of generality $k_1 \leq k_2$. If $k_1 = k_2$ and $x \neq y$ then x and y are incomparable in V and w_x and w_y are two distinct elements of $\mathcal{F}$ of the same size, hence are also incomparable in $\mathcal{F}$. So let us assume that $k_1 < k_2$ and $x \neq y$. Clearly since $|x| < |y|$, either $x > y$ (if $x \subsetneq y$) or x and y are incomparable. If $x \subsetneq y$ then by Obervation 3.1 $w_y \subsetneq w_x$ since $w_y \subseteq A_{i_{k_1}}$. If x and y are incomparable in V then there exists $z \in x \backslash y$, but since $x \subseteq X \subseteq A_{i_{k_2}}$ it must be the case that $z \in w_y$, and so since $z \notin w_x$, w_x and w_y must be incomparable. This shows that ψ preserves the order relation and finishes the proof of the claim. $\square$

By Claim 6.3 $\mathcal{F}$ contains an induced poset isomorphic to an element of $\mathcal{D}(|X|, m, \epsilon)$, and since $|X| \geq 2m$ it contains an induced copy of P by Lemma 2.2 and Lemma 2.1. This finishes the proof of Theorem 1.3 in the case where $a_t = m$.

Suppose now that $b_t = m$. Then there exist $i_0 < i_1 < \ldots < i_m$ such that for each k, $b_{i_k} = k$ and $b_{i_k} > b_{i_k-1}$, so that $\mathcal{S}_{i_k}$ is a set of k-anti-pivots for B_{i_k} in $(\mathcal{F}_{i_k-1})_{B_{i_k-1}, A_{i_k-1}}$. As above, it is easily seen that $\mathcal{S}_{i_k}$ is also a set of k-anti-pivots for B_{i_k} in $\mathcal{F}_{i_k-1}$.

Let $X = A_{i_m} \backslash B_{i_m}$; $|X| \geq 2m$ as before. For a fixed k, $0 \leq k \leq m$, let $V'_k = \mathcal{S}_{i_k} \cap X^{(k)}$, so V'_k is the set of k-anti-pivots of B_{i_k} which are contained in X. For $x \in V'_k$ let w_x be a witness of x being a k-anti-pivot of B_{i_k} in $\mathcal{F}_{i_k-1}$. Let $W'_k = \bigcup_{x \in V'_k} \{w_x\}$ and let $W' = \bigcup_{k=0}^{m} W'_k$. In a similar fashion as above, we have the following claim.

Claim 6.4. *W', viewed as a subposet of $\mathcal{F}$, is isomorphic to an element of $\mathcal{U}(|X|, m, \epsilon)$.*

Proof. Order the elements of V' so that $x \leq y$ if and only if $x \subseteq y$. Up to relabelling of the elements of X it is clear that $V' \in \mathcal{U}(|X|, m, \epsilon)$, since X is (ϵ, V'_k)-fat for every k, $0 \leq k \leq m$.

Let $\psi' : V' \longrightarrow W'$ be the map sending $x \in V'$ to $w_x \in W'$. We shall prove that ψ' is an isomorphism between $(V', \leq)$ and $(W', \subseteq)$, which proves the claim.

Suppose $x, y \in V'$ where $w_x \in \mathcal{F}_{i_k-1}$ for some i_{k_1} and $w_y \in \mathcal{F}_{i_{k_2}}$ for some i_{k_2} and without loss of generality $k_1 \leq k_2$. If $k_1 = k_2$ and $x \neq y$ then

x and y are incomparable in V' and w_x and w_y are two distinct elements of $\mathcal{F}$ of the same size, hence are also incomparable in $\mathcal{F}$. So let us assume that $k_1 < k_2$ and $x \neq y$. Clearly since $|x| < |y|$, either $x < y$ (if $x \subsetneq y$) or x and y are incomparable. If $x \subsetneq y$, then $w_x \subsetneq w_y$ by Observation 3.1 since $B_{i_{k_1}} \subseteq w_y$. If x and y are incomparable then there exists $z \in x \backslash y$, but since $z \in X \subseteq [n] \backslash B_{i_{k_2}}$, $z \notin w_y$, and so w_x and w_y must be incomparable. This shows that ψ' preserves the order relation and finishes the proof of the claim. □

By Claim 6.4 $\mathcal{F}$ contains an induced poset isomorphic to an element of $\mathcal{U}(|X|, m, \epsilon)$, and since $|X| \geq 2m$ it contains an induced copy of P by Lemma 2.2 and Lemma 2.1. This finishes the proof of Theorem 1.3 in the case where $b_t = m$. □

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